Reliably distinguishing states in qutrit channels using one-way LOCC

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We present numerical evidence showing that any three-dimensional subspace of $\mathbb{C}^3 \otimes \mathbb{C}^n$ has an orthonormal basis which can be reliably distinguished using one-way LOCC [1]. By measuring first in the environment, the conjecture would imply that the environment-assisted classical capacity of any rank three channel is at least $\log 3$. Similarly by measuring first on the system side, the conjecture would imply that the environment-assisting classical capacity of any qutrit channel is $\log 3$. We also show that one-way LOCC is not symmetric, by providing an example of a qutrit channel whose environment-assisted classical capacity is less than $\log 3$.

Background

It is known that any two orthonormal states in a bipartite space can be distinguished perfectly using LOCC [2]. We show that a related property holds for any three-dimensional subspace V in $\mathbb{C}^3 \otimes \mathbb{C}^n$, namely that there is an orthonormal basis which can be distinguished perfectly using LOCC, provided that we first perform a measurement on the $\mathbb{C}^3$ factor, and then use the result to select an optimal measurement on the $\mathbb{C}^n$ factor. When $\mathbb{C}^3$ is the system part and $\mathbb{C}^n$ is the environment, this is called an *environment-assisting* measurement, as opposed to an *environment-assisted* measurement in which a measurement is first performed on the environment and the result used to find an optimal measurement for the system. In general, we find that

$$N_{\{sys \longrightarrow env\}}(V) = 3, \quad N_{\{env \longrightarrow sys\}}(V) \leq 3$$

where N is the maximum number of orthonormal input states that can be reliably distinguished in the subspace V. This implies the channel capacity results stated above. Related work has been done in this area by Nathanson [3] and Fan [4].

Analysis

LOCC, or Local Operations and Classical Communication, is a method where local operations are performed separately on both parts of a bipartite system and information about the operations (such as the outcome of a measurement) are communicated classically between the parts.

Consider the following scenario. You wish to communicate a message using two states, $|\psi_1\rangle$ and $|\psi_2\rangle$, each in $\mathbb{C}^2 \otimes \mathbb{C}^2$. In order for the message to be successfully interpreted on the receiving end, the two states must be distinguished.

As an example, take the following states to be $|\psi_1\rangle$ and $|\psi_2\rangle$:

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

We are given $|\phi\rangle$ which is one of the above states. We choose the computational basis for our measurements; the measurement outcomes will be $|0\rangle$ or $|1\rangle$. Let's assume that we measure the first qubit, giving us the outcome $|0\rangle$. This means that the post-measurement state of $|\phi\rangle$ is one of the following two states:

$$|\psi_1'\rangle = |0\rangle$$
$$|\psi_2'\rangle = |1\rangle.$$

To determine which of these states it is, we measure the second qubit and get an outcome which is also $|0\rangle$. We now know that $|\phi\rangle = |\psi_1\rangle$, since only by measuring $|\psi_1\rangle$ can we possibly produce the outcomes $|00\rangle$. Table 1 in Figure 1 shows the correlations between outcomes and states for this scenario.

Now consider the case where the measurement outcome of the first qubit affects the choice of the optimal basis for the measurement of the second qubit:

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |1a_1\rangle)$$
$$|\psi_4\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |1a_2\rangle)$$

where $|a_i\rangle = \frac{1}{\sqrt{2}}|0\rangle \pm \frac{1}{\sqrt{2}}|1\rangle$. Distinguishing the two states requires that we first measure the first qubit to determine which basis to use to measure the second qubit. In other words, do we switch from the computational basis to the Hadamard basis for the second measurement, or do we measure again in the computational basis? Table 2 in Figure 1 shows the respective correlation between measurement outcomes and states.

1^{st}	2^{nd}		
0	0	$\Rightarrow$	$\lvert\psi_1\rangle$
0	1	$\Rightarrow$	$\lvert\psi_1\rangle$
1	0	$\Rightarrow$	$\lvert\psi_2\rangle$
1	1	$\Rightarrow$	$\lvert\psi_2\rangle$

1^{st}	2^{nd}		
0	0	$\Rightarrow$	$\lvert\psi_3\rangle$
0	1	$\Rightarrow$	$\lvert\psi_4\rangle$
1	a_1	$\Rightarrow$	$\lvert\psi_3\rangle$
1	a_2	$\Rightarrow$	$\lvert\psi_4\rangle$

FIGURE 1: Measurement outcomes on the input qubits and the deduced input state for the respective examples.

These protocols are examples of one-way LOCC. In $\mathbb{C}^2 \otimes \mathbb{C}^n$, it makes no difference in which order we measure the qubits.

The situation is almost analogous in $\mathbb{C}^3 \otimes \mathbb{C}^n$. Below is an example of three states in $\mathbb{C}^3 \otimes \mathbb{C}^3$ that can be distinguished using one-way LOCC.

$$|\psi_1\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |1a_1\rangle + |2b_1\rangle)$$
$$|\psi_2\rangle = \frac{1}{\sqrt{3}}(|01\rangle + |1a_2\rangle + |2b_2\rangle)$$
$$|\psi_3\rangle = \frac{1}{\sqrt{3}}(|02\rangle + |1a_3\rangle + |2b_3\rangle)$$

$|a_i\rangle, |b_j\rangle$ are orthogonal bases other than the computational basis in $\mathbb{C}^3$. It is clear that the choice of basis for the measurement of the second qutrit depends on the outcome of the measurement of the first qutrit.

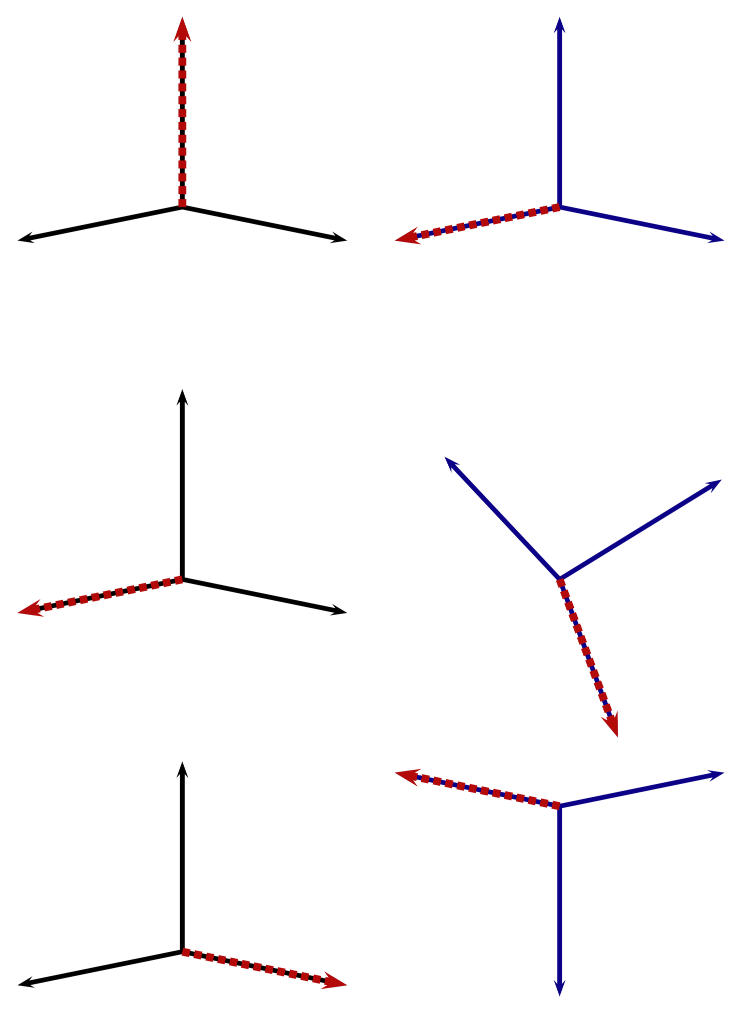

FIGURE 2: Graphical representation of sample measurement bases for the first and second qutrits for $|\psi_2\rangle$. The measurement outcomes for the first qutrit are $|0\rangle$ (top), $|1\rangle$ (middle) and $|2\rangle$ (bottom), and the measurement outcomes for the second qutrit pending the result of the first measurement, are $|1\rangle$, $|a_2\rangle$ and $|b_2\rangle$, respectively.

In general it is *not* possible to perfectly distinguish three orthogonal states in $\mathbb{C}^3 \otimes \mathbb{C}^n$ using LOCC. However we find that every three-dimensional subspace has an orthonormal basis which can be perfectly distinguished using one-way LOCC. The order in which the measurement is made is critical to being able to distinguish the two states: the first measurement must be done on the $\mathbb{C}^3$ part.

Asymmetry of LOCC

As mentioned above, one-way LOCC is asymmetric; we cannot always perfectly distinguish orthonormal bases in $\mathbb{C}^3 \otimes \mathbb{C}^n$ using environment-assisted measurements but not using environment-assisting measurements. An example of a subspace in $\mathbb{C}^3 \otimes \mathbb{C}^5$ which does not have a basis that can be reliably distinguished in such a way is shown below.

$$|\theta_1\rangle = \begin{pmatrix} -0.2450 - 0.0054i \\ -0.1694 + 0.0815i \\ 0.1071 - 0.3191i \\ 0.0655 - 0.3190i \\ -0.1911 - 0.1862i \\ 0.1185 + 0.3259i \\ -0.2530 + 0.0480i \\ 0.1194 - 0.1987i \\ 0.1948 - 0.2106i \\ 0.0595 + 0.2934i \\ 0.1286 - 0.1427i \\ -0.1420 + 0.1308i \\ -0.2367 + 0.1399i \\ 0.1384 - 0.0264 \\ 0.0867 + 0.1573i \end{pmatrix} \quad |\theta_2\rangle = \begin{pmatrix} 0.1438 + 0.2108i \\ -0.3214 + 0.1308i \\ 0.1229 + 0.0213i \\ 0.1775 - 0.1070i \\ 0.2091 - 0.1811i \\ -0.0937 + 0.1880i \\ 0.1609 + 0.0272i \\ 0.1705 + 0.0996i \\ -0.0630 + 0.0729i \\ 0.3389 - 0.1242i \\ 0.0201 - 0.2668i \\ 0.1127 - 0.3331i \\ 0.2338 + 0.3325i \\ -0.1798 - 0.0796i \\ -0.1097 + 0.1360i \end{pmatrix}$$

$$|\theta_3\rangle = \begin{pmatrix} 0.0390 - 0.0484i \\ 0.0405 - 0.2603i \\ 0.2206 + 0.2432i \\ -0.2843 - 0.0751 \\ -0.2416 - 0.1380i \\ 0.0510 + 0.3270i \\ 0.1691 + 0.0829i \\ -0.3761 - 0.1033i \\ 0.0988 + 0.1388i \\ 0.3138 + 0.2228i \\ 0.0553 + 0.2272i \\ 0.0468 - 0.0164i \\ 0.1966 - 0.1044i \\ -0.0147 + 0.1239i \\ -0.2313 + 0.0715i \end{pmatrix}$$

It is, however, unknown at this time *why* only environment-assisting, one-way LOCC is able to distinguish states perfectly.

Numerical Results

The goal is to find triplets of vectors in $\mathbb{C}^3 \otimes \mathbb{C}^n$ that can be distinguished perfectly. We pose the problem of distinguishing states as an optimization problem where the objective function that is minimized describes the mutual orthonormality of the environment states given that the system states are orthonormal. In other words, the algorithm seeks to find two bases, the first of which is guaranteed to be orthonormal, that allow the states to be distinguished. The only way this can occur is when the second basis is also orthogonal.

Description of Algorithm

Let $|\psi_i\rangle$ be three orthonormal states in $\mathbb{C}^{3n}$ which span the subspace, and let U be a parameterized unitary operator in $SU(3)$; U is thus subject to 8 optimizable parameters. Also, let $|e_i\rangle$ be a parameterized basis in $\mathbb{C}^3$, subject to an additional 8 optimizable parameters. Initially, we have

$$|\psi_1\rangle = |e_1A_1\rangle + |e_2B_1\rangle + |e_3C_1\rangle$$
$$|\psi_2\rangle = |e_1A_2\rangle + |e_2B_2\rangle + |e_3C_2\rangle$$
$$|\psi_3\rangle = |e_1A_3\rangle + |e_2B_3\rangle + |e_3C_3\rangle,$$

ignoring the amplitudes. First, U is applied to a matrix with columns $|\psi_i\rangle$, rotating the states, and thus searching for a better basis. Then, the basis $|e_i\rangle$ is rotated. States $|\psi_i\rangle$ are then projected onto $|e_i\rangle$, after which it is determined how mutually orthonormal the basis state triplets $|A_i\rangle$, $|B_i\rangle$ and $|C_i\rangle$ are. The mutual orthogonality of these three bases is minimized:

$$\min H = \sum_{i \neq j} |\langle A_i|A_j\rangle|^2 + |\langle B_i|B_j\rangle|^2 + |\langle C_i|C_j\rangle|^2.$$

Depending on the value of H, the parameters of U and $|e_i\rangle$ are accordingly adjusted and the process is repeated until H is minimized.

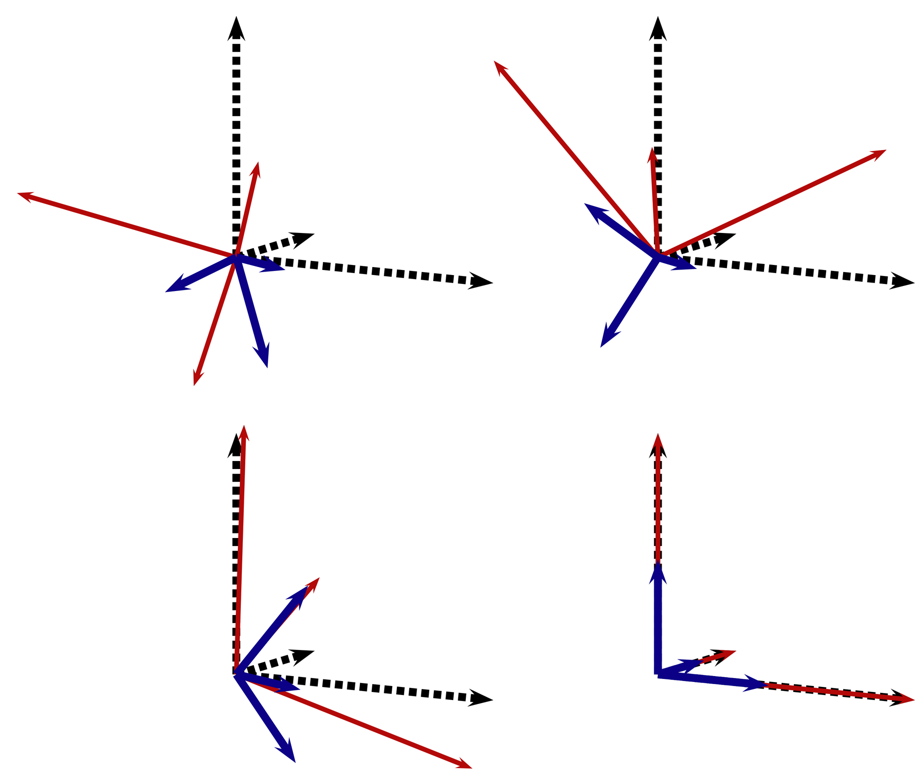

FIGURE 3: The states $|\psi_i\rangle$ (red vectors) in $\mathbb{C}^{3n}$ are rotated, as well as the basis $|e_i\rangle$ (blue vectors) which is also rotated, until a projection of $|\psi_i\rangle$ onto $|e_i\rangle$ results in three vectors (black dashed vectors) in $\mathbb{C}^n$ which are orthonormal.

Results

Since all spaces $\mathbb{C}^3 \otimes \mathbb{C}^n$, where $n \geq 3$, can be reduced to some Hilbert space $\mathbb{C}^3 \otimes \mathbb{C}^d$, where $3 \leq d \leq 9$, we only need to sample these seven subspaces. The table below gives the average minimum value found by the optimizer for each subspace, as well as the number of tested bases.

n	**Min.** H	**# subspaces**
3	2.816258×10^{-8}	138211
4	3.789893×10^{-8}	30271
5	3.789893×10^{-8}	32278
6	4.127063×10^{-8}	30000
7	4.394015×10^{-8}	30000
8	5.130496×10^{-8}	30216
9	5.594670×10^{-8}	30006

The optimization was successful in minimizing the objective function to zero for each sampled basis for the threshold of 10^{-6}.

Conclusion

This evidence leads us to believe that it is possible to perfectly distinguish any three orthogonal states in $\mathbb{C}^3 \otimes \mathbb{C}^n$ using one-way LOCC . Currently, it is unknown why it is possible to use coordinated partial product measurements in lieu of an entangled measurement. It is also unknown why the order of the partial measurements affects the degree to which states can be distinguished. Further investigation has also shown that one-way LOCC cannot be used to distinguish entangled bases in $\mathbb{C}^4 \otimes \mathbb{C}^n$ or higher.

This research was supported in part by National Science Foundation Grant DMS-0400426.

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